Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents

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The ocean rogue wave, one of the mysteries of nature, has not yet been understood or modelled satisfactorily, in spite of being in the intense lime-light in recent years and the concept spreading fast to other disciplines [1–9]. Rogue waves are extraordinarily high and steep surface waves. However, most of their theoretical models and experimental observations, excluding a few [3, 4, 10, 11] are one-dimensional, admitting limited high intensity and steepness. We propose here a novel two-dimensional integrable nonlinear Schrödinger equation allowing an exact lump-soliton with special asymmetry and directional preference. The soliton can appear on surface waves making a hole just before surging up high, with adjustable height and steepness and disappear again followed by the hole. The dynamics, speed and the duration of the soliton is controlled by ocean currents. These desirable properties make our exact model promising for describing deep sea large rogue waves.

RWs are reported to being observed in oceans, which apparently appear on a relatively calm sea from nowhere, make a sudden hole in the sea just before attaining surprisingly high amplitude and disappear again without a trace [8, 12, 13]. This elusive freak wave caught the imagination of the broad scientific community quite recently, triggering off an upsurge in theoretical [10, 11, 14, 15] and experimental [1–9] studies of this unique phenomenon. For identifying the class of such extreme waves the suggested signature of this rare event is a deviation of the probability distribution function (PDF) of the wave amplitude from its usual random Gaussian distribution (GD), by having a long-tail, indicating that the appearance of high intensity pulses are probable more often than that predicted by the GD [16]. In conformity with this definition RWs were detected in erbium doped fiber laser [9] in optical lasers [1, 8] , in nonlinear optical cavity [4] , in acoustic turbulence in He II [2] and other set ups [6]. As for the theoretical studies on the ocean RW, apart from few nonlinear models based on the modulation instability, four-wave mixing and conformal mapping [6, 15, 17] or on some other effects [4, 10] , the nonlinear Schrödinger (NLS) equation

$$i\partial_t q = \partial^2_{x^2} q + 2|q|^2 q, \tag{1}$$

is the most accepted one. Equation (1) is a well known integrable evolution equation in one space dimension (1D) admitting the Lax pair and exact soliton solutions [18]. Some models of RW generalise the NLS equation by adding extra terms on physical reasons like ocean current [13] , nonlinear dispersion [14, 19] etc., which however makes the system non-integrable, allowing only numerical simulations. The most popular RW model used in many studies is a unique analytic solution of the original equation (1), given by the Peregin breather (PB) or its trigonometric variant [5, 7, 11, 20, 21]. The conventional soliton solution: $q_s = \mathrm{sech}\kappa(x - vt)e^{i(kx+\omega t)}$ of the NLS equation (1), representing a localised translational wave behaves like a stable particle and unlike a RW propagates with unchanged shape and amplitude. In contrast, the exact PB solution of (1)

$$q_P(x, t) = e^{-2it}(u + iv),\ u = G - 1,\ v = -4tG, \tag{2}$$

where $G = 1/F,\ F(x, t) = x^2 + 4t^2 + \frac{1}{4}$, represents a breather mode $\cos 2t$ with unit intensity at both distant past and future, with its amplitude rising suddenly at $t = 0$ to its maximum at $x = 0$, but subsiding again with time to the same breathing state. This intriguing RW like behaviour makes the PB a popular candidate for the rogue wave [5, 7, 11, 21].

Notice however that, the NLS equation (1) together with its different generalisations are equations in $(1 + 1)$-dimensions and therefore all of their solutions, including the PB, can describe the time evolution of a wave only along a line. Looking more closely into the PB we also realise that the maximum amplitude attained by the RW described by this solution is fixed, and just three times more than that of the background waves. The steepness of this wave as well as the fastness of its appearance are also fixed, since solution (2) admits no free parameter. Therefore, though the well accepted PB or other solutions of the generalised NLS equation could be fitted into the working definition of the ocean RW, saying any wave with height more than twice the nearby *significant height* (average height among one-third of the highest waves) could be treated as the RW [16], they perhaps, with their severely restricted characteristics, can only explain moderately intense RW-like events in 1D, as observed in water channels [7], optical fibers [5, 9] or optical lasers [1, 8], but are grossly unfit as a model for the ocean RWs. First, ocean RWs are two-dimensional (2D) lumps appearing on the sea surface, which are not possible to describe by a 1D equation like the NLS (1). Second, the ocean RWs, as reported, might be as high as 17-30 meters in a relatively calm sea [8, 13], which is way above the value allowed by 1D solutions like the PB. Third, the steepness of the oceanic RW and the speed of their appearance may vary from event to event, whereas they are fixed in the PB (2) admitting no free parameter, and are of values much below than those reported for the deep sea RWs. Note that in 2D water basin experiments as well as in the related simulations the amplitude and steepness of the RWs were found to be higher [3, 6, 10, 11]

than those predicted and observed in 1D [7, 11].

The above arguments should be convincing enough to reject the PB together with other solutions of the generalised 1D NLS equation and look for a more realistic model for the ocean RWs. A straightforward 2D extension of the NLS equation:

$$i\partial_t q = d_1 \partial^2_{x^2} q - d_2 \partial^2_{y^y} q + 2|q|^2 q, \tag{3}$$

was proposed in some studies as a possible RW model [10, 11]. However the 2D NLS equation (3) is not an integrable system and gives only approximate solutions with no stable soliton. Nevertheless, this unlikely candidate is found to exhibit RW like structures numerically, with higher intensity and steepness and with an intriguing directional preference [3, 10] and broken special symmetry [4, 11].

In the light of not so satisfactory present state in modelling the deep sea RWs, we propose a new *integrable* extension of the 2D NLS equation:

$$i\partial_t q = d_1 \partial^2_{x^2} q - d_2 \partial^2_{y^2} q + 2iq(j_x - j_y), \; j_a = q\partial_a q^* - q^* \partial_a q \tag{4}$$

together with its exact lump-soliton as a suitable RW model. We find that, when the conventional *amplitude*-like nonlinear term in the non-integrable equation (3) is replaced by a *current*-like nonlinear term, the resulting equation (4) miraculously becomes a completely integrable system with all its characteristic properties, which is much rarer in 2D than in 1D. More surprisingly equation (4) admits an exact 2D generalisation of the PB with the desired properties of a real RW. Before proceeding further we notice that, our 2D NLS equation (4) can be simplified through a rotation on the plane to

$$i\partial_t q + \partial^2_{\bar{x}\bar{y}} q + 2iq(q\partial_{\bar{y}} q^* - q^* \partial_{\bar{y}} q), \tag{5}$$

where the *bar* over the coordinates will be omitted in what follows. Encouragingly, we can find an exact stable soliton solution of the 2D equation (5) as

$$q_{s(2d)}(x, y, t) = \mathrm{sech}\kappa(y + \rho x - vt) e^{k_1 x + k_2 y + \omega t}$$

together with the associated Lax pair, infinite set of conserved charges and higher soliton solutions proving its integrability (see *supplementary Note*). However, the most important finding relevant to our present problem, is to discover an exact solution of (5) as a 2D extension of the PB. Before presenting the dynamical solution we consider first its static 2D lump-like structure localised in both space directions, describing a fully developed RW:

$$q_{P(2d)}(x, y) \;=\; e^{4ix}(u + iv), \; u = G - 1, \; v = -4xG,$$
$$\text{where } G = \frac{1}{F}, \qquad F(x, y) = \alpha y^2 + 4x^2 + c. \tag{6}$$

One can check by direct insertion that (6) is an exact static solution of the 2D nonlinear equation (5) with two arbitrary parameters α and c. It marks an important difference of solution (6) from the well known PB (2), which has no free parameter in spite of its close resemblance with our solution. Looking closely into our 2D lump soliton (6), as shown in Fig. 1d, we find that the wave attains its maximum amplitude: $A_{rog} = (1-c)/c = N - 1$, at the centre $x = 0, \; y = 0$, where parameter $c = 1/N$ is chosen through an arbitrary integer N. On the other hand, at large distances: $|x| \to \infty, \; |y| \to \infty$ the wave goes to the background plane wave modulating as $\cos(4x)$, with its amplitude decreasing to a constant: $A_\infty = 1$, (Fig. 1a). The maximum amplitude relative to the background: $\frac{A_{rog}}{A_\infty} = N - 1$, reachable by our RW solution, can be adjusted by choosing N to fit different observed heights of the RWs, and particularly the deep sea RWs having significantly higher amplitudes [9, 13] than usually assumed [16]. Similarly, the steepness of the RW solution (6) at both sides as observed from the front: $|\partial_y q_{P(2d)}|$, linked to another free parameter α, can also be changed to fit different observations. Note that the amplitude of the wave falls to its minimum: $A_0 = 0$, at $x = 0, \; y = \pm y_0$ with $y_0 = \sqrt{\frac{1}{\alpha}(1-c)}$, which is relevant for the hole-wave formation as we see below.

Our next aim is to construct a dynamical lump soliton out of the static solution (6), to create a true picture of a RW which can appear and disappear fast with time. For constructing such a solution however we have to clarify first, whether it is possible in principle for our lump soliton to disappear, i.e. whether the soliton is free from all topological restrictions, which otherwise would prevent such a vanishing without a trace. The reason for such suspicion is due an interesting lesson from topology stating that, when a complex field $q(x, y)$ is defined on a 2D space with non-vanishing boundary condition $|q| \to 1$ at large distances, but having vanishing values $q \to 0$ close to the centre, we can define a unit vector $\hat{\phi} = \frac{q}{|q|}$ on an 1-sphere S^1. However, this vector $\hat{\phi} = (\phi_1, \phi_2)$ is well defined only at space boundaries: $\partial \mathrm{R}^2 \sim S^1$ (since $q = 0$ at inner points), realising a smooth map: $S^1 \to S^1$ with possible nontrivial topological charge $Q = n$. This charge with integer values $n = 0, 1, 2, \ldots$, labels the distinct homotopy classes and is defined as the degree of the map, which unlike a Nöther charge is conserved irrespective of the dynamics of the system. Such a situation occurs for example in type II superconductors with the charge linked to the quantised flux of vortices for the magnetic field $\mathbf{B}(x, y)$ [22] :

$$2\pi Q = \int d\mathbf{S} \cdot \mathbf{B} = \int_C \mathbf{dl} \cdot \mathbf{A}, \tag{7}$$

where $\mathbf{B} = curl\mathbf{A} = \partial_x \phi_1 \partial_y \phi_2 - \partial_x \phi_2 \partial_y \phi_1$. Notice that, our complex field solution $q_{P(2d)}(x, y)$ possesses clearly the features of $\hat{\phi}$ discussed above, since (6) goes to a constant modulation $-e^{4ix}$ at large distances and vanishes at points $(0, \pm y_0)$. Note that, such a solution related to

a sphere to sphere map can not go to a trivial configuration , if it belongs to a homotopy class with nontrivial topological charge: $Q = N, N = 1, 2, 3, ...,$ due to conservation of the charge, with the only exception for the class with zero charge $Q = 0$. Therefore, for confirming the possible appearance/disappearance property of a RW for solution (6), we have to establish first that in spite of defining a nontrivial topological map, it belongs nevertheless to the sector with topological charge: $Q = 0$, i.e. our lump soliton is indeed shrinkable to the *vacuum* solution. For this we calculate explicitly the topological charge (7) associated with (6) as

$$2\pi Q = \int_C \mathbf{dl} \cdot \mathbf{A} = \int (dxA_x + dyA_x), , \qquad (8)$$

with $A_a = \phi_1 \partial_a \phi_2, \ \phi_1 = Req/|q|, \ \phi_2 = Imq/|q|$, where the contour integral along x and y are taken along a closed square at the boundaries of the plane. Substituting explicit form of solution $q(x, y)$ from (6) and arguing about the oddness and evenness of the integrants with respect to x, y or checking directly by Mathematica one can show that the related charge is indeed $Q = 0$ and therefore the solution belongs to the trivial topological sector as we wanted. The intriguing reason behind this fact is that, the two holes appearing here have in fact opposite charges resulting to the combined charge being zero. For constructing a dynamical extension of the 2D RW solution (6) we realise that, the sudden change of amplitude with time, as necessary to mimic the RW behaviour, would result to a non-conservation of energy. Therefore it can not be described by an integrable equation alone, which insists on a strict conservation of all charges and therefore our integrable equation (5) needs certain modification. On the other hand, the importance of ocean currents in the formation of RWs, which is not considered in (5), is documented and repeatedly emphasised [13, 17] . This motivates us to solve both these problems in one go, by modifying our equation (5) with the inclusion of the effect of ocean currents, by adding a term in the form $-iU_c q_y$, as in [13] . For obtaining an exact dynamical RW solution to our modified 2D NLS equation, we choose the currents flowing along transverse directions and changing with the time and location as $U_c(y, t) = \frac{\mu t}{\alpha y}$. As is apparent, the currents would flow to the centre of formation of the RW from both the transverse directions, with the speed increasing as they approach, stopping however completely at the moment of the full surge, at $t = 0$. The picture gets reverted after the event with currents flowing back quickly, away from the centre. Though U_c looks ill-defined, the multiplicative factor q_y makes the current term well-behaved and the modified 2D NLS equation ((5) with the inclusion of the current term) admits now an exact dynamical 2D Peregin soliton, which interestingly has the same form as (6), only with the function F becoming dynamical with the inclusion of time variable as

$$F(x, y, t) = F(x, y) + \mu t^2, \quad F(x, y) = 4x^2 + \alpha y^2 + c. \quad (9)$$

The arbitrary parameter μ appearing in solution (9) is related to the ocean current and can control how fast the RW appears and how long it stays. It is convincingly demonstrated in Fig. 1 , how this exact dynamical 2D lump-soliton evolves from a background plane wave with a slight depression appearing already at the centre $x = 0, y = 0$ (Fig. 1a). As the time passes a sudden hole is formed right in the centre (Fig. 1b) at the moment $t_h = -\sqrt{\frac{1}{\mu}(1-c)}$, as told in marine-lore [11, 15], which splits into two and shift apart in the transverse direction (Fig. 1c) to make space for a high steep upsurge of the lump forming the actual RW (Fig. 1d) at time $t = 0$. The RW disappears fast into the background waves with the holes merging at the centre and vanishing again, thus describing vividly well the reported picture of the ocean RWs [9, 13, 17] as well as those found in large scale experiments [3]. The surface waves modelled by our solution (6), as visible also from Fig. 1, show a distict directional preference and an asymmetry between the two space variables.

a **b** **c** **d**

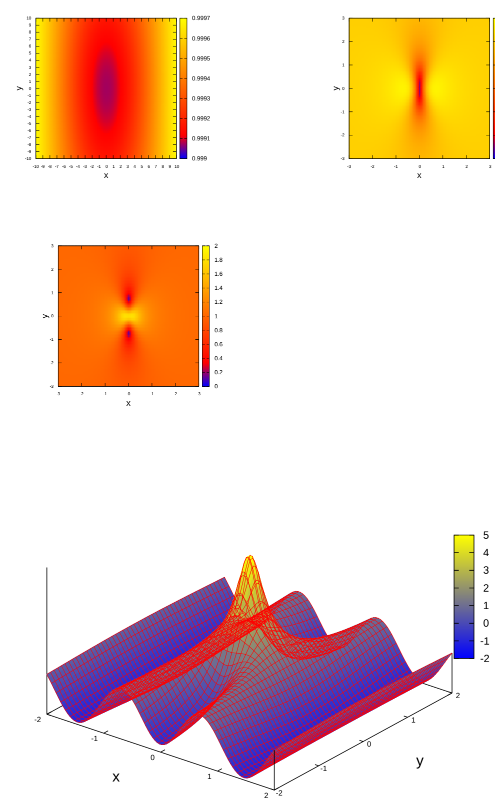

Figure 1: Formation of 2D rogue wave and holes Panels a-c show the modulus of the dynamical 2D lump-solution and panel d its real part at time $t = 0$. Figures correspond to parameter values $c = 1/6, \ \alpha = 1.2, \ \mu =$

1.2. **a,** Almost constant amplitude background wave with a slight depression appearing at the centre (at time $t = -30.00$). **b,** Creation of a hole at the centre at time $t_h = -0.83$. **c,** The hole splitted into two are drifting away from the centre (at $t = -0.40$). **d,** Full grown RW formed over the background modulation. The amplitude attained by the RW is five-times that of the background waves (linked to the choice of c).

We conclude by listing a few distinguishing features of our proposed dynamical lump soliton ((6) with (9)), which are important for a ocean RW model. 1) This is the first 2D dynamical RW model given in an analytic form. 2) It is an exact Peregin like lump soliton solution of a novel $1 + 2$-dimensional nonlinear integrable equation. 3) The dynamics of the solution is controlled by the ocean currents. 4) The height and steepness of the RW can be adjusted by independent free parameters. 5) The fastness of its appearance and the duration of stay can also be regulated by another parameter linked to the ocean current, which becomes the key factor in creating the dynamical RW. 6) The proposed solution exhibits a clear broken special symmetry in variables x and y as well as a directional preference, which are suspected to be crucial features in the formation of 2D RWs [3, 4, 10, 11]. 6) Strange appearance (and disappearance) of a hole just before (and after) the formation of the rogue wave [11, 17] is also confirmed in our model.

We hope that, this significant breakthrough in describing large ocean RWs through a dynamical analytic lump-soliton would also be valuable for experimental findings of 2D RWs in other systems like capillary fluid waves [6] optical cavity waves [4] and basin water waves [3].

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Supplementary Note: **Integrable properties of the Novel 2D NLS Equation** (equation (5) in the main text)

I. *Lax pair representation*:

The integrable nonlinear equations can be linked to a linear system

$$\Phi_x = U\Phi, \ \Phi_t = V\Phi, \tag{10}$$

associated with a pair of Lax martices U, V. The existence of a Lax pair is considered to be a guarantee for the integrability of a nonlinear partial differential equation. The Lax pair dependent on an additional parameter λ, called spectral parameter, are constructed in such a way that their flatness condition :

$$U_t - V_x + [U, V] = 0, \tag{11}$$

which is the compatibily condition of the overdetermined linear system, yields the given nonlinear equation [1].

For our nonlinear evolution equation (equation (5) in the main text) in 2D space we can find the Lax pair given through the standard 2×2 Pauli matrices $\sigma^a, a = +, -, 3$ as

$$U = i(U_{11}\sigma^3 + U_{12}\sigma^+ + U_{21}\sigma^-), \tag{12}$$

where $\quad U_{11} = 2\lambda^2 + |q|^2, U_{12} = U_{21}^* = 2\lambda q - iq_y$, and

$$V = i(V_{11}\sigma^3 + V_{12}\sigma^+ + V_{21}\sigma^-) \tag{13}$$

where

$$V_{11} = -4\lambda^3 + 2\lambda|q|^2 + i(qq_y^* - q^* q_y),$$

$$V_{12} = V_{21}^* = -4\lambda^2 q + 2i\lambda q_y + |q|^2 q + q_{yy},$$

where the notation $q_y = \partial_y q$ etc. as popular in the soliton commulity has been introduced. It can be checked directly that our integrable equation (5) (in the main text) is obtained from the flatness condition of the Lax pair (12, 13).

II. *Conserved charges & soliton solutions* :

The infinite set of conserved charges associated with our integrable system can be derived systematically from the Riccati equation obtained from the first equation in (10) with the Lax operator (12) [2]. Exact higher soliton solutions of equation (5) (in the main text) can be obtained by the Hirota's bilinearization method, similar to the NLS solitons [3].

III. *2D rogue waves as exact solutions* :

The static 2D rogue wave (6) (in the main text) as an exact solution of the 2D NLS equation (5) (in the main text) can be checked by direct insertion. The The dynamical 2D rogue wave solution (equation (6) with (9) in the main text) can be checked as an exact solution, by inserting it in the modified 2D NLS equation obtained by adding the ocean current term $-iU_c q_y$ to equation (5) (see the main text).

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